International Journal of Control, Energy and Electrical Engineering (CEEE)
Vol.7 pp.64-69

Robust MPC for fractional MIMO systems

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***Abstract*—This paper deals with fractional multi-input, multi-output systems that guarantees a very small number of parameters that can reduce the computation time. It focuses in particular on the state-space representation of systems which highlights the state variables and allows to study the internal behavior of the system taking into account the initial state. It also discusses the discretization of this type of system to finally adapt the Robust Model Predictive Control to apply it and shows its efficiency and performance in these systems.**

I. Introduction

The fractional systems proved their efficiency in the description of certain physical processes [1] [2]. It urged the researchers to study the behavior and the performances of the fractional systems [3] [4]. They also adapted several strategies of control for this type of system. The majority of the works which treat the fractional systems focus on SISO systems represented by transfer functions. Nevertheless, certain systems have several inputs or several outputs, that explains the necessity of studying the MIMO fractional systems. And in this case the state-space representation will be the best choice because it is easier to adapt to the systems MIMO. The rarity of searches and tools which treat the fractional MIMO systems represents one inconvenience of their use. For that reason this article focuses on the discrétisation and the control of this type of systems.

Model predictive control (MPC) strategy offers an effective way to tackle the problems in multivariable control system by including the process model in the computation of control actions [5]. For processes with strong interaction between different signals MPC can offer substantial performance improvement compared with traditional single-input single-output control strategies [6]. MPC has been used for several decades, and has been accepted as an important tool in many process industry applications. And from the control engineering viewpoint, MPC promises a great benefit to maintain the optimal economic operation of the plant and preserves the lifetime of the equipment. One of the main drawbacks of MPC is the difficulty to incorporate model uncertainties of plant explicitly, and for this reason, increasing attention has been placed on robust MPC problems.

This paper focuses on the state representation of MIMO fractional systems. It extends the discretization of fractional MIMO systems in the first section. In the second section it presents the robust predictive control that has been adapted for this type of systems. Simulation results are discussed in the third section.

II. Discritization of fractional state-space model

In the case of non-commensurate fractional systems, discretization must take into account the plurality of derivations of state variables, contrary to the commensurate case.

To move from a continuous model to a discrete model it is necessary to use this approximation [7], [8], [9]:

$$D^{\gamma}x(t)=\frac{1}{T_s^{\gamma}}\sum_{j=0}^{p}(-1)^j\binom{\gamma}{j}x((k-j)T_s) \tag{1}$$

Let's assume that the vector of continuous model derivation $\gamma=\begin{bmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_{nr}\end{bmatrix}^T$, T_s is the sampling time and $p\in\mathbb{N}$ is the number of samples with which the derivation was computed.

If $(i=1,\cdots,nr)$, the term $\binom{\gamma}{j}$ can be written as follows:

$$\binom{\gamma}{j}^T=\left[\binom{\gamma_1}{j}\quad\binom{\gamma_2}{j}\quad\cdots\quad\binom{\gamma_{nr}}{j}\right] \tag{2}$$

$$\binom{\gamma_i}{j}=\begin{cases}1 & for \quad j=0\\ \dfrac{\gamma_i(\gamma_i-1)\ldots(\gamma_i-j+1)}{j!} & for \quad j>0\end{cases} \tag{3}$$

By multiplying (1) by T_s^{γ} and developing the terms of $j=0$ and $j=1$ the found result is:

$$\begin{aligned}T_s^{\gamma}D^{\gamma}x(t)=&x(kT_s)-\gamma x((k-1)T_s)+\\&+\sum_{j=2}^{p}(-1)^j\binom{\gamma}{j}x((k-j)T_s)\end{aligned} \tag{4}$$

Let consider the following continuous fractional MIMO state-space model [10]:

$$\begin{cases}D^{\gamma}x(t) &= A_c x(t)+B_c u(t)\\ y(t) &= C_c x(t)\end{cases} \tag{5}$$

With $A_c\in\mathbb{R}^{nr\times nr}$, $B_c\in\mathbb{R}^{nr\times ni}$ and $C_c\in\mathbb{R}^{no\times nr}$ are the state matrices of the continuous fractional model and nr is the number of varibales in state-space model, ni is number of inputs and no is the number of outputs.

$$\begin{aligned}T_s^{\gamma}A_c x(kT_s)-x(kT_s)=&-\gamma x((k-1)T_s)+\\&+\sum_{j=2}^{p}(-1)^j\binom{\gamma}{j}x((k-j)T_s)-B_cT_su(kT_s)\end{aligned} \tag{6}$$

ISSN 2356-5608

note that $I_r \in \mathbb{R}^{nr\times nr}$ the identity matrix and T_s^{γ} the diagonal matrix filled in by ($T_s^{\gamma_1} \cdots T_s^{\gamma_{nr}}$).

To facilitate writing, let's note

$$\mathbb{Z} = (T_s^{\gamma} A_c - I_{nr})^{-1} \tag{7}$$

$$\begin{aligned} x(kT_s) = &-\mathbb{Z}\gamma x((k-1)T_s) + \mathbb{Z}\sum_{j=2}^{p}(-1)^j \binom{\gamma}{j} x((k-j)T_s) \\ &-\mathbb{Z}B_c T_s^{\gamma} u(kT_s) \end{aligned} \tag{8}$$

and with $(i = 1, \cdots, nr)$

$$c_j = diag\{(-1)^j \binom{\gamma_i}{j} \tag{9}$$

Now we can write the above equation as:

$$x(k) = \mathbb{Z}\, c_1 x(k-1) + \mathbb{Z}\sum_{j=2}^{p} c_j x(k-j) - \mathbb{Z}\, B_c T_s^{\gamma} u(k) \tag{10}$$

To simplify the equation:

$$A_j = \mathbb{Z}\, c_j \tag{11}$$

By expanding all terms and simplifying, the new form of (10)

$$x(k) = A_1 x(k-1) + A_2 x(k-2) + \cdots + A_k x(0) - \mathbb{Z} B_c T_s^{\gamma} u(k) \tag{12}$$

The system can therefore be described by a discrete state-space representation [11]:

$$\begin{cases} X_d(k+1) = A_d X_d(k) + B_d u(k) \\ y(k) = C_d X_d(k) \end{cases} \tag{13}$$

With

$$X_d(k+1) = \begin{pmatrix} x(k+1) \\ x(k) \\ \vdots \\ x(k-p+1) \end{pmatrix}, \quad X_d(k) = \begin{pmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-p) \end{pmatrix}$$

$$A_d = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix}, \quad B_d = \begin{pmatrix} -\mathbb{Z} B_c T_s^{\gamma} \\ 0_{ni} \\ \vdots \\ 0_{ni} \end{pmatrix}$$

$$\text{and } C_d = \begin{pmatrix} C & 0_{no} & \cdots & 0_{no} \end{pmatrix}$$

with u, y, X_d are respectively the input, output and variables state of the process. p is the number of past iterations which the system takes into account for calculating a variable, $0_{ni} \in \mathbb{R}^{1\times ni}$, $0_{no} \in \mathbb{R}^{no\times 1}$, $A_d \in \mathbb{R}^{(nr*p)\times(nr*p)}$, $B_d \in \mathbb{R}^{(nr*p)\times ni}$, $C_d \in \mathbb{R}^{no\times(nr*p)}$ and $X_d \in \mathbb{R}^{(nr*p)\times 1}$.

III. Robust Fractional MPC

The principle of the predictive control is to create an anticipatory effect for the system with respecting the trajectory to follow known in advance, based on the prediction of the future behavior of the system and minimizing the gap of these predictions to the trajectory and by minimizing a certain cost function J, within respecting operating constraints [12], [13].

In this section we have developed a predictive control from the discrete fractional state-space model described in the previous section. For that we will make a variable change : $\Delta X_d(k) = X_d(k) - X_d(k-1)$ the input variable difference: $\Delta u(k) = u(k) - u(k-1)$, and using it in (13) this transformation is found:

$$\Delta X_d(k+1) = A_d \Delta X_d(k) + B_d \Delta u(k) \tag{14}$$

The new state variable vector is:

$$X(k) = [\Delta X_d(k)^T \; y(k)]^T$$

with $y(k)$ is the output and :

$$y(k+1) - y(k) = C_d A_d \Delta X_d(k) + C_d B_d \Delta u(k) \tag{15}$$

The system can be written in the form:

$$\begin{cases} X(k+1) = AX(k) + B\Delta u(k) \\ y(k) = CX(k) \end{cases} \tag{16}$$

$$A = \begin{pmatrix} A_d & 0_d^T \\ C_d A_d & I_{no} \end{pmatrix}; \; B = \begin{pmatrix} B_d \\ C_d B_d \end{pmatrix};$$

$$C = \begin{pmatrix} 0_d & I_{no} \end{pmatrix}; \; 0_d \in \mathbb{R}^{no\times(nr*p)}$$

Future state variables can be predicted and written in the form:

$$\begin{cases} X(k+1) &= AX(k) + B\Delta u(k) \\ X(k+2) &= AX(k+1) + B\Delta u(k+1) \\ &= A^2 X(k) + AB\Delta u(k) + B\Delta u(k+1) \\ &\vdots \\ X(k+H_p) &= A^{H_p} X(k) + A^{H_p-1} B\Delta u(k) + \\ & \quad A^{H_p-2} B\Delta u(k+1) + \cdots \\ & \quad + A^{H_p-H_c} B\Delta u(k+H_c-1) \end{cases} \tag{17}$$

Based on (17) the future system outputs can be predicted:

$$\begin{cases} y(k+1) &= CAX(k) + CB\Delta u(k) \\ y(k+2) &= CAX(k+1) + CB\Delta u(k+1) \\ &= CA^2 X(k) + CAB\Delta u(k) + \\ & \quad CB\Delta u(k+1) \\ &\vdots \\ y(k+H_p) &= CA^{H_p} X(k) + CA^{H_p-1} B\Delta u(k) + \\ & \quad CA^{H_p-2} B\Delta u(k+1) + \cdots \\ & \quad + CA^{H_p-H_c} B\Delta u(k+H_c-1) \end{cases} \tag{18}$$

H_p and H_c are respectively the prediction horizon and the control horizon with $H_p \geq H_c$. Assume the vector Y

ISSN 2356-5608

which contains H_p system's predicted future outputs and Δu contains H_c future controls:

$$Y^T = [y(k+1)\ \ y(k+2)\cdots y(k+H_p)]$$

$$\Delta u^T = [\Delta u(k)\,\Delta u(k+1)\cdots\Delta u(k+H_c-1)]$$

The vector Y can also be written as :

$$Y = FX(k) + \Phi\Delta u \tag{19}$$

$$F = \begin{pmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{H_p} \end{pmatrix} \tag{20}$$

$$\Phi^T = \begin{pmatrix} CB & CAB & CA^2B & \cdots & CA^{H_p-1}B \\ 0 & CB & CAB & \cdots & CA^{H_p-2}B \\ 0 & 0 & CB & \cdots & CA^{H_p-3}B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & CA^{H_p-H_c}B \end{pmatrix} \tag{21}$$

The aim of predictive control is to find the control vector Δu which forces the system's output y to follow the setpoint y_s. In order to achieve this we must optimize a criterion J which represents the control objective [14]:

$$J = \sum_{i=1}^{H_p}(y_s(k+i) - y(k+i))^2 + \lambda\sum_{i=0}^{H_c-1}\Delta u^2(k+i) \tag{22}$$

The criterion J can be written in matrix form:

$$J = (Y_s - Y)^T(Y_s - Y) + \Delta u^T\lambda\Delta u \tag{23}$$

With $Y_s^T = [y_s(k+1)\ \ y_s(k+2)\cdots y_s(k+H_p)]$ is the vector filled by the future values of the set-points and λ is weight coefficient on the control.

Let consider in the following, a state-space description of an uncertain system that can generally be written in the form [15] [16]:

$$\begin{cases} D^\gamma x(t) &= A_c(\theta)x(t) + B_c(\theta)u(t) \\ y(t) &= C_c(\theta)x(t) \end{cases} \tag{24}$$

With A_c, B_c and C_c are the state matrices of the continuous fractional model, the vector of continuous model derivation $\gamma = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_{nr}]^T$ and nr is the number of varibales in state-space model.

$$A_c(\theta)) = A_{c0} + \Delta A_c \tag{25}$$

$$\Delta A_c = \sum_{i=1}^{N}\theta_i A_{ci} \tag{26}$$

with

$$\theta_i \in \left[\underline{\theta}_i, \overline{\theta}_i\right]$$

For this representation, matrices A_{ci} distribute the uncertainty on the different elements of the matrix $A_c(\theta)$.

In RFMPC the control sequence represents the best solution to the worst case. Consequently, the optimal control law can be obtained by the resolution of the following min-max problem [17]

$$\min_{\Delta u}\quad \max_{\theta}\quad J(\Delta u, \theta) \tag{27}$$

The min-max problem is resolved in two steps. The first step consists to calculate the maximum of the performance criterion $J(\Delta u, \theta)$ compared to the uncertainties parameters of the set θ . Starting with an initial solution, RFMPC searches the solution of following function in taking into account constraints on the parameters model.

$$J^*(\Delta u) = \max_{\theta}\quad J(\Delta u, \theta) \tag{28}$$

The second step concerns the minimization of the criterion $J^*(\Delta u, \theta^*)$ in taking into account the solution found in Eq. (28) and the control sequence constraints:

$$J_2 = \min_{\Delta u}\quad J^*(\Delta u) \tag{29}$$

IV. Simulation results

Consider a fractional MIMO system whose state-space representation is of the form:

$$A_c = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad C_c = \begin{pmatrix} 1.25 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 1.3 \\ 0.9 \end{pmatrix}$$

with $a_1 = 1.25$ and $a_2 = 0.625$

The step response of system is shown in Fig.1.

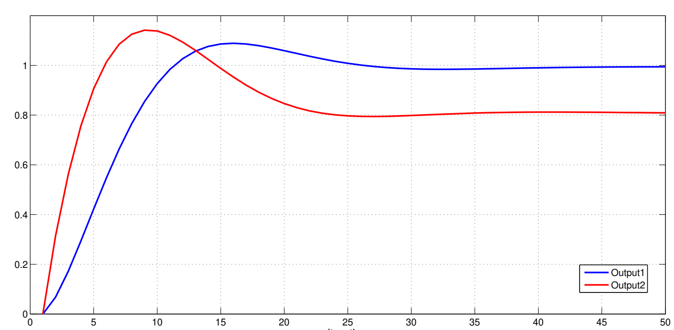

Fig. 1. The step responnse of system.

Let consider in this section that the system is represented by the Oustaloup model called M equivalent to the previous state-space, during the simulation the variables a_2 will change to 0.425 and 0.825 to find respectively the models of Oustaloup M_1 and M_2.

For each simulation the real system will be represented as follows:

- for $1 \leq k \leq 50$: the system is the model M.
- for $51 \leq k \leq 100$: the system is the model M_1.
- for $101 \leq k \leq 160$: the system is the model M_2.

The chosen predictive control parameters are: H_p=3, H_c=1, λ=1. The chosen sampling period for discretization is: $T_s = 0.3s$.

To ensure a better control for the system an uncertainty will be imposed on the values of $a_1 = 1.25$ and $a_2 = 0.625$ with :

$$a_1 \in [1, 1.5] \text{ and } a_2 \in [0.375, 0.875]$$

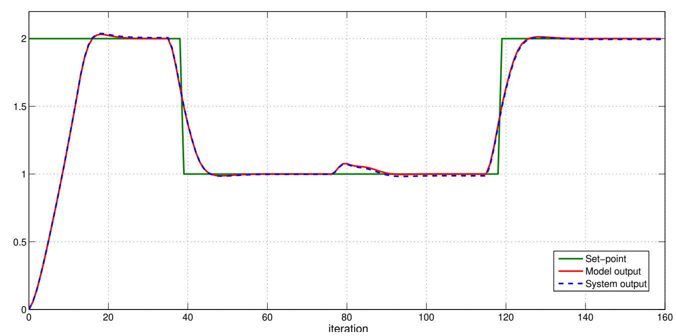

Fig. 2. First output.

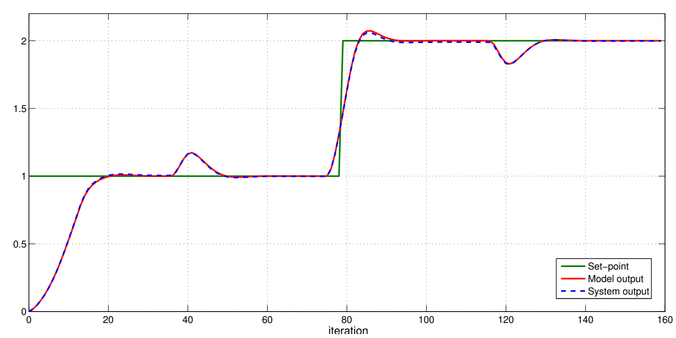

Fig. 3. Second output.

Fig.2 and Fig.3 show that RFMPC is able to force the system to follow the set-point. At each iteration RFMPC find the worst values of a_1 and a_2 for the system and then calculate the best value of control to satisfy the criterion J. The worst values of Δa_1 and Δa_2 are shown in Fig.4.

Fig.5 and Fig.6 represent the signals of control generated by RFMPC.

The choosen constraint on the control variables is $\| \Delta u(k) \| \leq 0.2$.

Assuming that the control variable $\Delta u(k)$ can only increases or decreases in a unit of magnitude less than 0.2 [18], the operational constraint is :

$$-0.2 \leq \Delta u(k) \leq 0.2$$

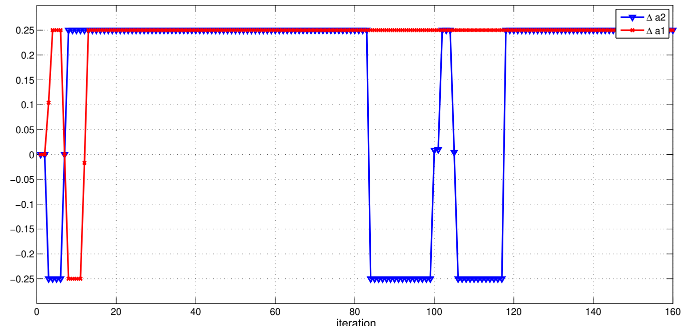

Fig. 4. The worst values of Δa_1 and Δa_2 in each iteration.

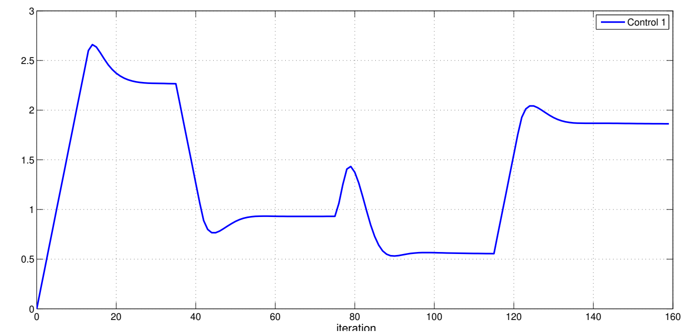

Fig. 5. First control .

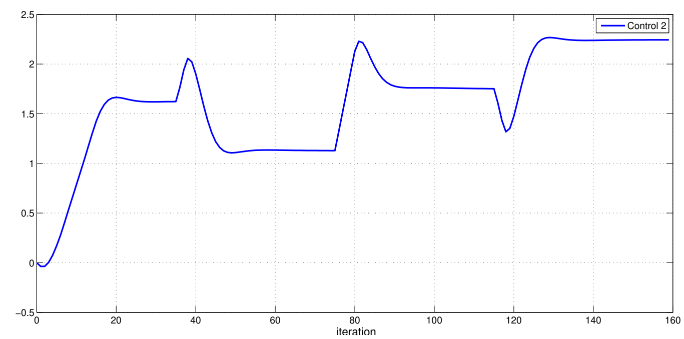

Fig. 6. Second control.

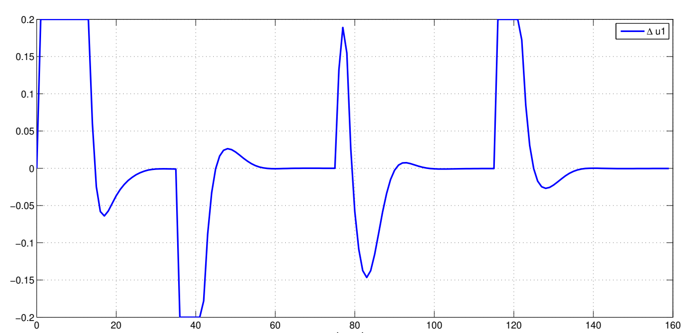

Fig. 7. Increment of first control.

Even under constraint the RFMPC can ensure that the outputs follow the set-point. Constrains on the control

ISSN 2356-5608

International Journal of Control, Energy and Electrical Engineering (CEEE)
Vol.7 pp.64-69

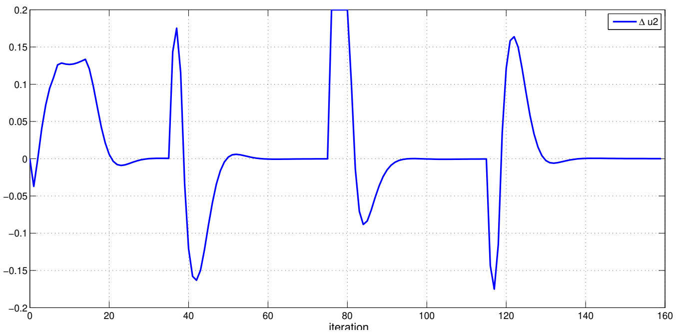

Fig. 8. Increment of second control.

increment represented in Fig.7 and Fig.8 guarantees that there is no peaks in the control signal. In return, the outputs porsuite becomes slower as shown in the Fig.2 and Fig.3. The choice of the the interval Δu is very important, because if the interval is too wide the condition will not be taken into account when minimizing criterion J, and if the interval is too small the control will no longer be able to bring the outputs to follow the set-point, even if it happen the system will be too slow.

V. Conclusion

The use of fractional models becomes more and more frequent given the efficiency they provide in the discription of certain physical systems. Nevertheless they remain a little difficult to handle. The majority of research [19], [20], [21] deals with SISO fractional models that have proved effective at describing several physical phenomena. This article has adapted predictive control to apply it to a MIMO fractional system. The importance of this work is that it deals with the state-space representation of MIMO fractional systems from discritization to control. It first introduced the dicritization of fractional MIMO state-space representation. Then for the same kind of system, it adapted the robust predictive control and applied it.

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ISSN 2356-5608